# Examples of public goods decision processes, given logarithmic valuation

## Definitions

Suppose that there are $N$ individuals, indexed by $i$ or $j = 1, \ldots, N$. Each individual has preferences for private consumption $x_i$ and a public good $y$, defined by the following quasi-linear utility function:

$$U_i = x_i + \alpha_i \ln y$$

$$MU_i = \frac{\alpha_i}{y}$$

Let $p$ be the (constant) marginal cost of providing the public good, so that the total cost is $py$.

## Pareto optimal provision of $y$

For the amount of $y$ to be optimal, the marginal social benefit, equal to the sum of individual marginal benefits, should be just equal to the marginal cost of the public good.

$$\sum_i MU_i = \frac{\sum_i \alpha_i}{y} \overset{\text{set}}{=} p$$

$$\boxed{y^o = \frac{\sum_i \alpha_i}{p} = \frac{N}{p}\alpha_{mean}}$$

## Provision with no coordination

If there is no coordination, then once the marginal utility of the highest public good demander is equal to the marginal cost, no one will have an incentive to individually provide another unit of the public good.

$$MU_{max} = \frac{\alpha_{max}}{y} \overset{\text{set}}{=} p$$

$$\boxed{y^* = \frac{1}{p}\alpha_{max}}$$

Notice that $y^o / y^* = N\alpha_{mean} / \alpha_{max}$; i.e. they can differ greatly if $N$ is large.

**Voting equilibrium with equal division**

If the cost of the public good is divided equally, so that each person pays $py/N$, heterogeneous preferences will lead to different voter ideal points $y_i^*$. Because preferences over $y$ are single-peaked, there will be a majority voting equilibrium at the ideal point of the median voter.

$$MU_i = \frac{\alpha_i}{y} \overset{\text{set}}{=} \frac{p}{N}$$

$$y_i^* = \frac{N}{p}\alpha_i$$

$$\boxed{y^* = \frac{N}{p}\alpha_{median}}$$

Note that, if the mean value of $\alpha$ is equal to the median value, the equilibrium reached by voting is Pareto efficient.

**Lindahl scheme**

If utility functions are known by a benevolent planner, he can assign cost shares to citizens so that they will unanimously prefer the Pareto optimal quantity of $y$ to all other possible quantities. To find the optimal shares, we find the voter ideal points as a function of their shares, set these ideal points equal to the Pareto optimal value $y^o$, and solve for $s_i$.

$$MU_i = \frac{\alpha_i}{y} \overset{\text{set}}{=} s_i p$$

$$y_i^* = \frac{\alpha_i}{s_i p} \overset{\text{set}}{=} y^o = \frac{\sum_j \alpha_j}{p}$$

$$s_i^o = \frac{\alpha_i}{\sum_j \alpha_j}$$

An alternative interpretation of the Lindahl shares is as the outcome of an idealized negotiation process, in which there are no bargaining costs / impediments, so that all potential Pareto gains are realized.

**Lindahl equilibrium with self-reported utility**

Suppose that the benevolent planner can’t discern the individuals’ utilities, but rather asks them to report their values of $\alpha$, and then proceeds to decide the public good quantity $y$ and the cost shares $s_i$ with the assumption that the reported values are true. Define $A_i$ as person $i$'s reported value of his own $\alpha_i$.

$$y = \frac{\sum_i A_i}{p} \qquad\qquad s_i = \frac{A_i}{\sum_j A_j}$$

Define $A_{\sim i}$ as $\sum_{j \neq i} A_j$, i.e. the sum of $A$ values reported by people other than $i$. Thus, $\sum_j A_j = A_i + A_{\sim i}$, and the terms above can be rewritten as

$$y = \frac{A_i + A_{\sim i}}{p} \qquad\qquad s_i = \frac{A_i}{A_i + A_{\sim i}}$$

Given that $\omega_i$ is person $i$'s starting wealth, his utility is given by

$$U_i = \omega_i - s_i p y + \alpha_i \ln y$$

$$U_i = \omega_i - \frac{A_i}{A_i + A_{\sim i}} p \frac{A_i + A_{\sim i}}{p} + \alpha_i \ln\left(\frac{A_i + A_{\sim i}}{p}\right)$$

$$U_i = \omega_i - A_i + \alpha_i \ln\left(\frac{A_i + A_{\sim i}}{p}\right)$$

Person $i$'s choice variable is his reported value $A_i$. We can find his optimal choice $A_i^*$ by setting $\partial U_i / \partial A_i = 0$:

$$\frac{\partial U_i}{\partial A_i} = -1 + \frac{\alpha_i}{\left(\frac{A_i + A_{\sim i}}{p}\right)} \frac{1}{p} \overset{\text{set}}{=} 0$$

$$\frac{\alpha_i}{A_i + A_{\sim i}} = 1$$

$$\boxed{A_i^* = \alpha_i - A_{\sim i}}$$

We assume that people are not allowed to report negative $A_i$'s; therefore, if $A_{\sim i} > \alpha_i$, person $i$'s best response is to report $A_i = 0$. In the Nash equilibrium, it must be true that

$$\sum_i A_i^* = \alpha_{max}$$

Plugging this into the formula used to determine $y$, we find that the equilibrium is equivalent to the equilibrium without coordination:

$$\boxed{y^* = \frac{1}{p} \alpha_{max}}$$